

A COMPARISON OF q -DECOMPOSITION NUMBERS IN THE q -DEFORMED FOCK SPACES OF HIGHER LEVELS

KAZUTO IIJIMA

ABSTRACT. The q -deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado. The q -decomposition matrix is a transition matrix from the standard basis to the canonical basis defined by Uglov in the q -deformed Fock space. In this paper, we show that parts of q -decomposition matrices of level ℓ coincides with that of level $\ell - 1$ under certain conditions of multi charge.

1. INTRODUCTION

The q -deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado [JMMO91]. For a multi charge $s = (s_1, \dots, s_\ell) \in \mathbb{Z}^\ell$, the q -deformed Fock space $F_q[s]$ of level ℓ is the $\mathbb{Q}(q)$ -vector space whose basis are indexed by ℓ -tuples of Young diagrams. i.e. $\{|\lambda; s\rangle \mid \lambda \in \Pi^\ell\}$, where Π is the set of Young diagrams. Heisenberg group (resp. quantum group $U_q(\hat{\mathfrak{sl}}_n)$) acts on $F_q[s]$ as level $q^{n\ell}$ (resp. level q^ℓ). Both actions commute on $F_q[s]$.

The canonical bases $\{G^+(\lambda; s) \mid \lambda \in \Pi^\ell\}$ and $\{G^-(\lambda; s) \mid \lambda \in \Pi^\ell\}$ are bases of the Fock space $F_q[s]$ that are invariant under a certain involution $-$ [Ugl00]. Define matrices $\Delta^+(q) = (\Delta_{\lambda,\mu}^+(q))_{\lambda,\mu}$ and $\Delta^-(q) = (\Delta_{\lambda,\mu}^-(q))_{\lambda,\mu}$ by

$$G^+(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^+(q) |\mu; s\rangle \quad , \quad G^-(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^-(q) |\mu; s\rangle.$$

We call $\Delta_{\lambda,\mu}^+(q)$ and $\Delta_{\lambda,\mu}^-(q)$ q -decomposition numbers. These q -decomposition matrices play an important role in representation theory. However it is difficult to compute q -decomposition matrices.

In the case of $\ell = 1$, Varagnolo-Vasserot [VV99] proved that $\Delta^+(1)$ coincides with the decomposition matrix of v -Schur algebra. Ariki defined a q -analogue of decomposition numbers of v -Schur algebra by using Khovanov-Lauda's grading, and proved that it coincides with the q -decomposition numbers [Ari]. For $\ell \geq 2$, Yvonne [Yvo06] conjectured that the matrix $\Delta^+(q)$ coincides with the q -analogue of the decomposition matrices of cyclotomic Schur algebras at a primitive n -th root of unity under a suitable condition on multi charge.

Let $O_s(\ell, 1, m)$ be the category \mathcal{O} of rational Cherednik algebra of $(\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_m$ associated with multicharge s . Rouquier [Rou08, Theorem 6.8, §6.5] conjectured that, for arbitrary multi charge, the multiplicities of simple modules in standard modules in $O_s(\ell, 1, m)$ are equal to the corresponding coefficients $\Delta_{\lambda,\mu}^+(q)$, where $m = |\lambda| = |\mu|$. It is expected that $\oplus_{m \geq 0} O_s(\ell, 1, m)$ should categorify $F_1[s]$. (see [Sha] for the details.) More generally, it is expected that, together with a suitable grading, $\oplus_{m \geq 0} O_s(\ell, 1, m)$ should categorify $F_q[s]$. For the detail of correspondence between the charges of $O_s(\ell, 1, m)$ and the charges of Fock spaces, see [Rou08].

Now, we state our main theorems. We say that the j -th component s_j of the multi charge is *sufficiently large* for $|\lambda; s\rangle$ if $s_j - s_i \geq \lambda_1^{(i)}$ for any $i = 1, 2, \dots, \ell$, and that s_j is *sufficiently small* for $|\lambda; s\rangle$ if $s_i - s_j \geq |\lambda| = |\lambda^{(1)}| + \dots + |\lambda^{(\ell)}|$ for any $i = 1, 2, \dots, \ell$ (see Definition 3.1). More generally, for a positive integer N we say that s_j is sufficiently small for N if $s_i - s_j \geq N$ for all $i \neq j$. If s_j is sufficiently large

for $|\lambda; s\rangle$ and $|\lambda; s\rangle > |\mu; s\rangle$, then the j -th components of λ and μ are both the empty Young diagram \emptyset (Lemma 3.2). On the other hand, if s_j is sufficiently small for $|\lambda; s\rangle$ and $|\lambda; s\rangle \geq |\mu; s\rangle$, then $\mu^{(j)} = \emptyset$ implies $\lambda^{(j)} = \emptyset$. (Lemma 3.3).

Our main results are as follows:

Theorem A. (Theorem 3.4)

Let $\varepsilon \in \{+, -\}$. If s_j is sufficiently large for $|\lambda; s\rangle$, then

$$\Delta_{\lambda, \mu; s}^\varepsilon(q) = \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^\varepsilon(q),$$

where $\check{\lambda}$ (resp. $\check{\mu}, \check{s}$) is obtained by omitting the j -th component of λ (resp. μ, s), $\Delta_{\lambda, \mu; s}^\varepsilon(q)$ is the q -decomposition number of level ℓ and $\Delta_{\check{\lambda}, \check{\mu}; \check{s}}^\varepsilon(q)$ is the q -decomposition number of level $\ell - 1$.

Theorem B. (Theorem 3.5)

Let $\varepsilon \in \{+, -\}$. If s_j is sufficiently small for $|\mu; s\rangle$ and $\mu^{(j)} = \emptyset$, then

$$\Delta_{\lambda, \mu; s}^\varepsilon(q) = \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^\varepsilon(q),$$

where $\check{\lambda}$ (resp. $\check{\mu}, \check{s}$) is obtained by omitting the j -th component of λ (resp. μ, s).

Shoji and Wada proved some product formulae of q -decomposition numbers [SW09, Theorem 2.9]. There are some overlaps between our results and their product formula. [SW09] has some assumptions “dominance” on the multi charge while our results don’t. On the concluding facts, [SW09] has a flexibility of embedding of q -decomposition matrices while our results don’t.

Our results are related to category \mathcal{O} in the following sense. In the category \mathcal{O} , Chuang and Miyachi conjectured the following:

Conjectures. [CM, §5]

(A’) Let $\lambda' \in \Pi^\ell$. If s_1 is sufficiently large for any $|(\emptyset, \lambda'); s\rangle$, there exists an embedding

$$\mathcal{O}_{\check{s}}(\ell, 1, m) \hookrightarrow \mathcal{O}_s(\ell + 1, 1, m).$$

(B’) If s_ℓ is sufficiently small for m , there exists a quotient functor

$$\mathcal{O}_s(\ell + 1, 1, m) \twoheadrightarrow \mathcal{O}_{\check{s}}(\ell, 1, m),$$

where \check{s} is obtained by omitting the j -th component of s .

We see that Conjecture (A’) (resp. (B’)) is consistent with Theorem A (resp. Theorem B) by taking into account the conjecture that $\oplus_{m \geq 0} \mathcal{O}_s(\ell, 1, m)$ should categorify $F_q[s]$. Theorem A (resp. Theorem B) gives a strong support to the conjecture (A’) (resp. (B’)).

This paper is organized as follows. In Section 2, we review the q -deformed Fock spaces of higher levels and its canonical bases. In Section 3, we state the main results. In Section 4, we review the straightening rules in the q -deformed Fock spaces. Theorem A(Theorem 3.4) and Theorem B(Theorem 3.5) are proved in Section 5 and 6 respectively.

Acknowledgments. I am deeply grateful to Hyohe Miyachi and Soichi Okada for their advice.

Notations. For a positive integer N , a *partition* of N is a non-increasing sequence of non-negative integers summing to N . We write $|\lambda| = N$ if λ is a partition of N . The *length* $l(\lambda)$ of λ is the number of non-zero components of λ . And we use the same notation λ to represent the Young diagram corresponding to λ . For an ℓ -tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)})$ of Young diagrams, we put $|\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}| + \dots + |\lambda^{(\ell)}|$.

2. THE q -DEFORMED FOCK SPACES OF HIGHER LEVELS

2.1. q -wedge products and straightening rules. Let n, ℓ, s be integers such that $n \geq 2$ and $\ell \geq 1$. We define $P(s)$ and $P^{++}(s)$ as follows;

- (1) $P(s) = \{\mathbf{k} = (k_1, k_2, \dots) \in \mathbb{Z}^\infty \mid k_r = s - r + 1 \text{ for any sufficiently large } r\}$
 (2) $P^{++}(s) = \{\mathbf{k} = (k_1, k_2, \dots) \in P(s) \mid k_1 > k_2 > \dots\}.$

Let Λ^s be the $\mathbb{Q}(q)$ vector space spanned by the q -wedge products

$$(3) \quad u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \dots, \quad (\mathbf{k} \in P(s))$$

subject to certain commutation relations, so-called straightening rules. Note that the straightening rules depend on n and ℓ . [Ugl00, Proposition 3.16] (The precise description will be given in §4.)

Example 2.1. (i) For every $k_1 \in \mathbb{Z}$, $u_{k_1} \wedge u_{k_1} = -u_{k_1} \wedge u_{k_1}$. Therefore $u_{k_1} \wedge u_{k_1} = 0$.

(ii) Let $n = 2$, $\ell = 2$, $k_1 = -2$, and $k_2 = 4$. Then

$$u_{-2} \wedge u_4 = q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0.$$

(iii) Let $n = 2$, $\ell = 2$, $k_1 = -1$, $k_2 = -2$ and $k_3 = 4$. Then

$$\begin{aligned} u_{-1} \wedge u_{-2} \wedge u_4 &= u_{-1} \wedge (u_{-2} \wedge u_4) = u_{-1} \wedge (q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0) \\ &= q u_{-1} \wedge u_4 \wedge u_{-2} + (q^2 - 1) u_{-1} \wedge u_2 \wedge u_0 \end{aligned}$$

By applying the straightening rules, every q -wedge product $u_{\mathbf{k}}$ is expressed as a linear combination of so-called *ordered q -wedge products*, namely q -wedge products $u_{\mathbf{k}}$ with $\mathbf{k} \in P^{++}(s)$. The ordered q -wedge products $\{u_{\mathbf{k}} \mid \mathbf{k} \in P^{++}(s)\}$ form a basis of Λ^s called *the standard basis*.

2.2. Abacus. It is convenient to use the abacus notation for studying various properties in straightening rules.

Fix an integer $N \geq 2$, and form an infinite abacus with N runners labeled $1, 2, \dots, N$ from left to right. The positions on the i -th runner are labeled by the integers having residue i modulo N .

$$\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ -N+1 & -N+2 & \cdots & -1 & 0 \\ 1 & 2 & \cdots & N-1 & N \\ N+1 & N+2 & \cdots & 2N-1 & 2N \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Each $\mathbf{k} \in P^{++}(s)$ (or the corresponding q -wedge product $u_{\mathbf{k}}$) can be represented by a bead-configuration on the abacus with $n\ell$ runners and beads put on the positions k_1, k_2, \dots . We call this configuration *the abacus presentation* of $u_{\mathbf{k}}$.

Example 2.2. If $n = 2$, $\ell = 3$, $s = 0$, and $\mathbf{k} = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \dots)$, then the abacus presentation of $u_{\mathbf{k}}$ is

$d = 1$		$d = 2$		$d = 3$		
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
$\textcircled{-17}$	$\textcircled{-16}$	$\textcircled{-15}$	$\textcircled{-14}$	$\textcircled{-13}$	$\textcircled{-12}$	$\cdots m = 3$
$\textcircled{-11}$	$\textcircled{-10}$	$\textcircled{-9}$	$\textcircled{-8}$	$\textcircled{-7}$	-6	$\cdots m = 2$
$\textcircled{-5}$	$\textcircled{-4}$	-3	$\textcircled{-2}$	-1	0	$\cdots m = 1$
$\textcircled{1}$	$\textcircled{2}$	$\textcircled{3}$	4	5	$\textcircled{6}$	$\cdots m = 0$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
$c = 1$	$c = 2$	$c = 1$	$c = 2$	$c = 1$	$c = 2$	

We use another labeling of runners and positions. Given an integer k , let c, d and m be the unique integers satisfying

$$(4) \quad k = c + n(d - 1) - n\ell m, \quad 1 \leq c \leq n \quad \text{and} \quad 1 \leq d \leq \ell.$$

Then, in the abacus presentation, the position k is on the $c + n(d - 1)$ -th runner (see the previous example). Relabeling the position k by $c - nm$, we have ℓ abaci with n runners.

Example 2.3. In the previous example, relabeling the position k by $c - nm$, we have

$d = 1$		$d = 2$		$d = 3$		
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
$\textcircled{-5}$	$\textcircled{-4}$	$\textcircled{-5}$	$\textcircled{-4}$	$\textcircled{-5}$	$\textcircled{-4}$	$\cdots m = 3$
$\textcircled{-3}$	$\textcircled{-2}$	$\textcircled{-3}$	$\textcircled{-2}$	$\textcircled{-3}$	-2	$\cdots m = 2$
$\textcircled{-1}$	0	-1	0	-1	0	$\cdots m = 1$
$\textcircled{1}$	$\textcircled{2}$	$\textcircled{1}$	2	1	$\textcircled{2}$	$\cdots m = 0$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
$c = 1$	$c = 2$	$c = 1$	$c = 2$	$c = 1$	$c = 2$	

We assign to each of ℓ abacus presentations with n runners a q -wedge product of level 1. In fact, straightening rules in each “sector” are the same as those of level 1 by identifying the abacus in the sector with that of level 1. (see also [Ugl00] and §4.1 for the detail)

We introduce some notation.

Definition 2.4. For an integer k , let c, d and m be the unique integers satisfying (4), and write

$$(5) \quad u_k = u_{c-nm}^{(d)}.$$

Also we write $u_{c_1-nm_1}^{(d_1)} > u_{c_2-nm_2}^{(d_2)}$ if $k_1 > k_2$, where $k_i = c_i + n(d_i - 1) - n\ell m_i$, ($i = 1, 2$).

We regard $u_{c-nm}^{(d)}$ as u_{c-nm} in the case of $\ell = 1$.

Example 2.5. If $n = 2$, $\ell = 3$, then we have

$$u_{-10} \wedge u_1 = -q^{-1} u_1 \wedge u_{-10} + (q^{-2} - 1) u_{-4} \wedge u_{-5},$$

that is,

$$u_{-2}^{(1)} \wedge u_1^{(1)} = -q^{-1} u_1^{(1)} \wedge u_{-2}^{(1)} + (q^{-2} - 1) u_0^{(1)} \wedge u_{-1}^{(1)}.$$

On the other hand, in the case of $n = 2, \ell = 1$,

$$u_{-2} \wedge u_1 = -q^{-1} u_1 \wedge u_{-2} + (q^{-2} - 1) u_0 \wedge u_{-1}.$$

2.3. ℓ -tuples of Young diagrams. Another indexation of the ordered q -wedge products is given by the set of pairs (λ, s) of ℓ -tuples of Young diagrams $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ and integer sequences $s = (s_1, \dots, s_\ell)$ summing up to s . Let $\mathbf{k} = (k_1, k_2, \dots) \in P^{++}(s)$, and write

$$k_r = c_r + n(d_r - 1) - n\ell m_r, \quad 1 \leq c_r \leq n, \quad 1 \leq d_r \leq \ell, \quad m_r \in \mathbb{Z}.$$

For $d \in \{1, 2, \dots, \ell\}$, let $k_1^{(d)}, k_2^{(d)}, \dots$ be integers such that

$$\beta^{(d)} = \{c_r - nm_r \mid d_r = d\} = \{k_1^{(d)}, k_2^{(d)}, \dots\} \quad \text{and} \quad k_1^{(d)} > k_2^{(d)} > \dots$$

Then we associate to the sequence $(k_1^{(d)}, k_2^{(d)}, \dots)$ an integer s_d and a partition $\lambda^{(d)}$ by

$$k_r^{(d)} = s_d - r + 1 \quad \text{for sufficiently large } r \quad \text{and} \quad \lambda_r^{(d)} = k_r^{(d)} - s_d + r - 1 \quad \text{for } r \geq 1.$$

In this correspondence, we also write

$$(6) \quad u_{\mathbf{k}} = |\lambda; s\rangle \quad (\mathbf{k} \in P^{++}(s)).$$

Example 2.6. If $n = 2, \ell = 3, s = 0$, and $\mathbf{k} = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \dots)$, then

$$\begin{aligned} k_1 &= 6 = 2 + 2(3 - 1) - 6 \cdot 0, & k_2 &= 3 = 1 + 2(2 - 1) - 6 \cdot 0, \\ k_3 &= 2 = 2 + 2(1 - 1) - 6 \cdot 0, & \dots & \text{and so on.} \end{aligned}$$

Hence,

$$\beta^{(1)} = \{2, 1, 0, -1, -2, \dots\}, \quad \beta^{(2)} = \{1, 0, -2, -3, -4, \dots\}, \quad \beta^{(3)} = \{2, -3, -4, -5, \dots\}.$$

Thus, $s = (2, 0, -2)$ and $\lambda = (\emptyset, (1, 1), (4))$.

Note that we can read off $s = (2, 0, -2)$ and $\lambda = (\emptyset, (1, 1), (4))$ from the abacus presentation. (see Example 2.3)

2.4. The q -deformed Fock spaces of higher levels.

Definition 2.7. For $s \in \mathbb{Z}^\ell$, we define the q -deformed Fock space $F_q[s]$ of level ℓ to be the subspace of Λ^s spanned by $|\lambda; s\rangle$ ($\lambda \in \Pi^\ell$):

$$(7) \quad F_q[s] = \bigoplus_{\lambda \in \Pi^\ell} \mathbb{Q}(q) |\lambda; s\rangle.$$

We call s a multi charge.

2.5. The bar involution.

Definition 2.8. The involution $\overline{}$ of Λ^s is the \mathbb{Q} -vector space automorphism such that $\overline{q} = q^{-1}$ and

$$(8) \quad \overline{u_{\mathbf{k}}} = \overline{u_{k_1} \wedge \dots \wedge u_{k_r} \wedge u_{k_{r+1}} \wedge \dots} = (-q)^{\kappa(d_1, \dots, d_r)} q^{-\kappa(c_1, \dots, c_r)} (u_{k_r} \wedge \dots \wedge u_{k_1}) \wedge u_{k_{r+1}} \wedge \dots,$$

where c_i, d_i are defined by k_i as in (4), r is an integer satisfying $k_r = s - r + 1$. And $\kappa(a_1, \dots, a_r)$ is defined by

$$\kappa(a_1, \dots, a_r) = \#\{(i, j) \mid i < j, a_i = a_j\}.$$

Remarks (i) The involution is well defined. i.e. it doesn't depend on r [Ugl00].

(ii) The involution comes from the bar involution of affine Hecke algebra \hat{H}_r . (see §4 for more detail.)

(iii) The involution preserves the q -deformed Fock space $F_q[s]$ of higher level.

2.6. The dominance order. We define a partial ordering $|\lambda; s\rangle \geq |\mu; s\rangle$. For $|\lambda; s\rangle$ and $|\mu; s\rangle$, we define multi-sets $\tilde{\lambda}$ and $\tilde{\mu}$ as

$$\begin{aligned}\tilde{\lambda} &= \{\lambda_a^{(d)} + s_d \mid 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}, \\ \tilde{\mu} &= \{\mu_a^{(d)} + s_d \mid 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}.\end{aligned}$$

We denote by $(\tilde{\lambda}_1, \tilde{\lambda}_2, \dots)$ (resp. $(\tilde{\lambda}_1, \tilde{\lambda}_2, \dots)$) the sequence obtained by rearranging the elements in the multi-set $\tilde{\lambda}$ (resp. $\tilde{\mu}$) in decreasing order.

Definition 2.9. Let $|\lambda; s\rangle = u_{k_1} \wedge u_{k_2} \wedge \dots$ and $|\mu; s\rangle = u_{g_1} \wedge u_{g_2} \wedge \dots$. We define $|\lambda; s\rangle \geq |\mu; s\rangle$ if $|\lambda| = |\mu|$ and

$$(9) \quad \begin{cases} (a) & \tilde{\lambda} \neq \tilde{\mu} \quad , \quad \sum_{j=1}^r \tilde{\lambda}_j \geq \sum_{j=1}^r \tilde{\mu}_j \quad (\text{for all } r = 1, 2, 3, \dots) \quad , \text{ or} \\ (b) & \tilde{\lambda} = \tilde{\mu} \quad , \quad \sum_{j=1}^r k_j \geq \sum_{j=1}^r g_j \quad (\text{for all } r = 1, 2, 3, \dots) \quad . \end{cases}$$

Remark. The order in Definition 2.9 is different from the order in [Ugl00] (see Example 2.10 below). However, the unitriangularity in (11) holds for both of them.

Example 2.10. Let $n = \ell = 2$, $s = (1, -1)$, $\lambda = ((1, 1), \emptyset)$, and $\mu = (\emptyset, (2))$. Then, $|\lambda; s\rangle = u_2 \wedge u_1 \wedge u_{-1} \wedge u_{-3} \wedge \dots$ and $|\mu; s\rangle = u_3 \wedge u_1 \wedge u_{-2} \wedge u_{-3} \wedge \dots$. In Uglov's order, $|\mu; s\rangle$ is greater than $|\lambda; s\rangle$. However, $|\lambda; s\rangle > |\mu; s\rangle$ under our order since $\{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\} = \{2, 2, -1\}$ and $\{\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3\} = \{1, 1, 1\}$.

We define a matrix $(a_{\lambda, \mu}(q))_{\lambda, \mu}$ by

$$(10) \quad \overline{|\lambda; s\rangle} = \sum_{\mu} a_{\lambda, \mu}(q) |\mu; s\rangle.$$

Then the matrix $(a_{\lambda, \mu}(q))_{\lambda, \mu}$ is unitriangular with respect to \geq , that is

$$(11) \quad \begin{cases} (a) & \text{if } a_{\lambda, \mu}(q) \neq 0, \text{ then } |\lambda; s\rangle \geq |\mu; s\rangle, \\ (b) & a_{\lambda, \lambda}(q) = 1. \end{cases}$$

(see the identity (27) for the detail.)

Thus, by the standard argument, the unitriangularity implies the following theorem.

Theorem 2.11. [Ugl00] There exist unique bases $\{G^+(\lambda; s) \mid \lambda \in \Pi^\ell\}$ and $\{G^-(\lambda; s) \mid \lambda \in \Pi^\ell\}$ of $F_q[s]$ such that

$$\begin{aligned} (i) \quad & \overline{G^+(\lambda; s)} = G^+(\lambda; s) \quad , \quad \overline{G^-(\lambda; s)} = G^-(\lambda; s) \\ (ii) \quad & G^+(\lambda; s) \equiv |\lambda; s\rangle \pmod{q \mathcal{L}^+} \quad , \quad G^-(\lambda; s) \equiv |\lambda; s\rangle \pmod{q^{-1} \mathcal{L}^-} \\ \text{where} \quad & \mathcal{L}^+ = \bigoplus_{\lambda \in \Pi^\ell} \mathbb{Q}[q] |\lambda; s\rangle \quad , \quad \mathcal{L}^- = \bigoplus_{\lambda \in \Pi^\ell} \mathbb{Q}[q^{-1}] |\lambda; s\rangle. \end{aligned}$$

Definition 2.12. Define matrices $\Delta^+(q) = (\Delta_{\lambda,\mu}^+(q))_{\lambda,\mu}$ and $\Delta^-(q) = (\Delta_{\lambda,\mu}^-(q))_{\lambda,\mu}$ by

$$(12) \quad G^+(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^+(q) |\mu; s\rangle \quad , \quad G^-(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^-(q) |\mu; s\rangle.$$

The entries $\Delta_{\lambda,\mu}^{\pm}(q)$ are called *q-decomposition numbers*. Note that *q-decomposition numbers* $\Delta^{\pm}(q)$ depend on n, ℓ and s . The matrices $\Delta^+(q)$ and $\Delta^-(q)$ are also unitriangular with respect to \geq .

It is known [Ugl00, Theorem 3.26] that the entries of $\Delta^-(q)$ are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type A , and that they are polynomials in q with non-negative integer coefficients (see [KT02]).

3. A COMPARISON OF q -DECOMPOSITION NUMBERS

3.1. Sufficiently large and sufficiently small.

Definition 3.1. Let $s = (s_1, s_2, \dots, s_{\ell}) \in \mathbb{Z}^{\ell}$ be a multi charge and $1 \leq j \leq \ell$.

(i). We say that the j -th component s_j of the multi charge s is *sufficiently large* for $|\lambda; s\rangle \in F_q[s]$ if

$$(13) \quad s_j - s_i \geq \lambda_1^{(i)} \quad \text{for all } i = 1, 2, \dots, \ell.$$

More generally, we say that s_j is sufficiently large for a q -wedge u_k if

$$(14) \quad s_j \geq c_r - nm_r \quad \text{for all } r = 1, 2, \dots,$$

where $k_r = c_r + n(d_r - 1) - n\ell m_r$, ($r = 1, 2, \dots$), $1 \leq c \leq n$ and $1 \leq d \leq \ell$ (see §2).

(ii). We say that s_j is *sufficiently small* for $|\lambda; s\rangle$ if

$$(15) \quad s_i - s_j \geq |\lambda| = |\lambda^{(1)}| + \dots + |\lambda^{(\ell)}| \quad \text{for all } i \neq j.$$

Note that the definition of sufficiently small depends only on the size of λ and the multi charge s . When we fix the multi charge s , we say that s_j is *sufficiently small* for N if

$$(16) \quad s_i - s_j \geq N \quad \text{for all } i \neq j.$$

Remark. If $|\lambda; s\rangle$ is 0-dominant in the sense of [Ugl00], that is

$$s_i - s_{i+1} \geq |\lambda| = |\lambda^{(1)}| + \dots + |\lambda^{(\ell)}| \quad \text{for all } i = 1, 2, \dots, \ell - 1,$$

then s_1 is sufficiently large for $|\lambda; s\rangle$ and s_{ℓ} is sufficiently small for $|\lambda; s\rangle$.

Lemma 3.2. If s_j is sufficiently large for $|\lambda; s\rangle$ and $|\lambda; s\rangle \geq |\mu; s\rangle$, then

(i) $\lambda^{(j)} = \emptyset$,

(ii) s_j is also sufficiently large for $|\mu; s\rangle$. In particular, $\mu^{(j)} = \emptyset$.

Proof. It is clear that $\lambda^{(j)} = \emptyset$ by the definition.

Note that

$$\begin{aligned} s_j \text{ is sufficiently large for } |\lambda; s\rangle &\Leftrightarrow s_j - s_i \geq \lambda_1^{(i)} \quad \text{for all } i = 1, 2, \dots, \ell \\ &\Leftrightarrow s_j \geq \max\{\lambda_1^{(1)} + s_1, \dots, \lambda_1^{(\ell)} + s_{\ell}\} = \tilde{\lambda}_1. \end{aligned}$$

If $|\lambda; s\rangle \geq |\mu; s\rangle$, then $\tilde{\lambda}_1 \geq \tilde{\mu}_1$ and so $s_j \geq \tilde{\mu}_1$. It means that s_j is sufficiently large for $|\mu; s\rangle$. \square

Lemma 3.3. *Suppose that s_j is sufficiently small for $|\lambda; s\rangle$. If $|\lambda; s\rangle \geq |\mu; s\rangle$ and $\mu^{(j)} = \emptyset$, then $\lambda^{(j)} = \emptyset$.*

Proof. Suppose that $l(\lambda^{(j)}) \geq 1$. Then s_j is the minimal integer in the set $\{\mu_a^{(d)} + s_d \mid 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}$ because $\mu^{(j)} = \emptyset$ and s_j is the minimal integer in s . On the other hand, the minimal integer in the set $\{\lambda_a^{(d)} + s_d \mid 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}$ is greater than s_j because s_j is sufficiently small for $|\lambda; s\rangle$. Therefore $|\lambda; s\rangle \not\geq |\mu; s\rangle$. This is a contradiction. \square

3.2. Main results. Now, we are ready to state our main theorems. We will prove the theorems in §5 and §6 respectively.

Theorem 3.4. *Let $\varepsilon \in \{+, -\}$. If s_j is sufficiently large for $|\lambda; s\rangle$, then*

$$(17) \quad \Delta_{\lambda, \mu; s}^\varepsilon(q) = \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^\varepsilon(q),$$

where $\check{\lambda}$ (resp. $\check{\mu}, \check{s}$) is obtained by omitting the j -th component of λ (resp. μ, s).

Theorem 3.5. *Let $\varepsilon \in \{+, -\}$. If s_j is sufficiently small for $|\mu; s\rangle$ and $\mu^{(j)} = \emptyset$, then*

$$(18) \quad \Delta_{\lambda, \mu; s}^\varepsilon(q) = \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^\varepsilon(q),$$

where $\check{\lambda}$ (resp. $\check{\mu}, \check{s}$) is obtained by omitting the j -th component of λ (resp. μ, s).

Example 3.6. (i) If $n = \ell = 2$, $s = (3, -3)$ and $\lambda = (\emptyset, (6))$, $\mu = (\emptyset, (5, 1))$, then s_1 is sufficiently large for $|\lambda; s\rangle$. Hence

$$\Delta_{\lambda, \mu; s}^-(q) = \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) = \Delta_{(6), (5, 1); (-3)}^-(q) = -q^{-1}.$$

(ii) If $n = \ell = 2$, $s = (3, -3)$ and $\lambda = ((6), \emptyset)$, $\mu = ((5, 1), \emptyset)$, then s_2 is sufficiently small for $|\mu; s\rangle$. Hence

$$\Delta_{\lambda, \mu; s}^-(q) = \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) = \Delta_{(6), (5, 1); (-3)}^-(q) = -q^{-1}.$$

4. Q-WEDGES AND STRAIGHTENING RULES

In this section, we review the straightening rules [Ugl00] to prove our main results.

4.1. affine Hecke algebra and straightening rules. In this paragraph, we review the affine Hecke algebra of type A_1 and straightening rules. We treat only the case of type A_1 . Indeed for our proof we only need the straightening rule of q -wedge whose length is equal to two. In fact, the straightening rules for a q -wedge which length is greater than 2 is obtained from the straightening rules for two adjacent element. (see Example 5.4) More general case, see [Ugl00].

The Hecke algebra H of type A_1 is the algebra over $\mathbb{Q}(q)$ with generator T_1 and relation

$$(19) \quad (T_1 - q^{-1})(T_1 + q) = 0.$$

The affine Hecke algebra \hat{H} is the tensor space H and the polynomial ring $\mathbb{Q}(q)[X_1^\pm, X_2^\pm]$ with relations

$$(20) \quad X^\lambda T_1 = T_1 X^{s_1(\lambda)} + (q - q^{-1}) \frac{X^{s_1(\lambda)} - X^\lambda}{1 - X_1 X_2^{-1}},$$

$$(21) \quad T_1 X^\lambda = X^{s_1(\lambda)} T_1 + (q - q^{-1}) \frac{X^{s_1(\lambda)} - X^\lambda}{1 - X_1 X_2^{-1}},$$

where $\lambda \in \mathbb{Z}^2$ and s_1 is the transposition.

Let P_1 be the $\mathbb{Q}(q)$ -vector space whose basis is $\{ (c_1, c_2) \mid c_1, c_2 \in \mathbb{Z}, 1 \leq c_1 \leq n, 1 \leq c_2 \leq n \}$. Define the right action of H on P_1 as

$$(22) \quad (c_1, c_2) \cdot T_1 = \begin{cases} (c_2, c_1) & \text{if } c_1 < c_2, \\ q^{-1} (c_1, c_1) & \text{if } c_1 = c_2, \\ (c_2, c_1) - (q - q^{-1})(c_1, c_2) & \text{if } c_1 > c_2. \end{cases}$$

Let P_2 be the $\mathbb{Q}(q)$ -vector space whose basis is $\{ |d_1, d_2\rangle \mid d_1, d_2 \in \mathbb{Z}, 1 \leq d_1 \leq \ell, 1 \leq d_2 \leq \ell \}$. Define the left action of H on P_2 as

$$(23) \quad T_1 \cdot |d_1, d_2\rangle = \begin{cases} |d_2, d_1\rangle & \text{if } d_1 < d_2, \\ -q |d_1, d_1\rangle & \text{if } d_1 = d_2, \\ |d_2, d_1\rangle - (q - q^{-1}) |d_1, d_2\rangle & \text{if } d_1 > d_2. \end{cases}$$

Define a vector space Λ by

$$\Lambda = P_1 \otimes_H \hat{H} \otimes_H P_2.$$

Definition 4.1. [Ugl00] For $(c_1, c_2) \in P_1$, $|d_1, d_2\rangle \in P_2$, and $m_1, m_2 \in \mathbb{Z}$, put $k_j = c_j + n(d_j - 1) - n\ell m_j$, ($j = 1, 2$). Denote $(c_1, c_2) \otimes X_1^{m_1} X_2^{m_2} \otimes |d_1, d_2\rangle \in \Lambda$ by

$$(24) \quad u_{k_1} \wedge u_{k_2}.$$

Proposition 4.2. [Ugl00] For integers k_1, k_2 , let c_j, d_j, m_j be the unique integers satisfying $k_j = c_j + n(d_j - 1) - n\ell m_j$, $1 \leq c_j \leq n$ and $1 \leq d_j \leq \ell$, ($j = 1, 2$). Then,

$$(25) \quad u_{k_2} \wedge u_{k_1} = (-q^{-1})^{\delta_{d_1=d_2}} \left\{ q^\alpha u_{k_1} \wedge u_{k_2} + \text{sgn}(m) (q - q^{-1}) \sum_{j=\beta}^{|m_1-m_2|-\gamma} u_{k_1-c_1+c_2-\text{sgn}(m)n\ell j} \wedge u_{k_2+c_1-c_2+\text{sgn}(m)n\ell j} \right\},$$

where

$$\text{sgn}(m) = \begin{cases} 1 & \text{if } m_1 < m_2 \\ -1 & \text{if } m_1 > m_2 \\ 0 & \text{if } m_1 = m_2 \end{cases}, \quad \alpha = \begin{cases} 1 & \text{if } c_1 = c_2 \text{ and } k_1 > k_2 \\ -1 & \text{if } c_1 = c_2 \text{ and } k_1 < k_2 \\ 0 & \text{if } c_1 \neq c_2 \end{cases},$$

$$\delta_{d_1=d_2} = \begin{cases} 1 & \text{if } d_1 = d_2 \\ 0 & \text{if } d_1 \neq d_2 \end{cases}, \quad \beta = \begin{cases} 0 & \text{if } c_1 > c_2, m_1 < m_2 \text{ or } c_1 < c_2, m_1 > m_2 \\ 1 & \text{if otherwise} \end{cases},$$

and

$$\gamma = \begin{cases} 1 & \text{if } d_1 < d_2, m_1 < m_2 \text{ or } d_1 > d_2, m_1 > m_2 \\ 0 & \text{if } d_1 > d_2, m_1 < m_2 \text{ or } d_1 < d_2, m_1 > m_2 \end{cases}.$$

Proof. We only show the statement in the case $c_1 = c_2$, $m_1 < m_2$, and $d_1 < d_2$. The other case can be treated similarly.

In this case, $k_1 > k_2$, $\delta_{d_1=d_2} = 0$, $\text{sgn}(m) = 1$, $\alpha = 1$, $\beta = 1$, and $\gamma = 1$. Note that $X_1 X_2$ and T_1 commute each other thanks to the relation (20), that is $X_1 X_2 T_1 = T_1 X_1 X_2$. From the relation (20), for any positive integer N we have

$$(26) \quad X_1^N T_1 = T_1^{-1} X_2^N + (q - q^{-1})(X_1 X_2^{N-1} + X_1^2 X_2^{N-2} + \cdots + X_1^{N-1} X_2).$$

Hence

$$\begin{aligned} & u_{k_2} \wedge u_{k_1} \\ &= (c_1, c_1 | \otimes X_1^{m_2} X_2^{m_1} \otimes |d_2, d_1) \quad (\text{by Definition 4.1}) \\ &= (c_1, c_1 | \otimes (X_1 X_2)^{m_1} X_1^{m_2-m_1} T_1 \otimes |d_1, d_2) \quad (\text{by (23)}) \\ &= (c_1, c_1 | \otimes (X_1 X_2)^{m_1} \left\{ T_1^{-1} X_2^{m_2-m_1} + (q - q^{-1})(X_1 X_2^{m_2-m_1-1} + X_1^2 X_2^{m_2-m_1-2} + \cdots + X_1^{m_2-m_1-1} X_2) \right\} \otimes |d_1, d_2) \\ & \quad (\text{by (26)}) \\ &= q(c_1, c_1 | \otimes X_1^{m_1} X_2^{m_2} \otimes |d_1, d_2) + (q - q^{-1})(c_1, c_1 | \otimes (X_1^{m_1+1} X_2^{m_2-1} + X_1^{m_1+2} X_2^{m_2-2} + \cdots + X_1^{m_2-1} X_2^{m_1+1}) \otimes |d_1, d_2) \\ & \quad (\text{by (22)}) \\ &= q u_{k_1} \wedge u_{k_2} + (q - q^{-1}) \left(u_{k_1-n\ell} \wedge u_{k_2+n\ell} + u_{k_1-2n\ell} \wedge u_{k_2+2n\ell} + \cdots + u_{k_1-n\ell(m_2-m_1-1)} \wedge u_{k_2+n\ell(m_2-m_1-1)} \right) \\ & \quad (\text{by Definition 4.1}) \\ &= q u_{k_1} \wedge u_{k_2} + (q - q^{-1}) \sum_{j=1}^{m_2-m_1-1} u_{k_1-n\ell j} \wedge u_{k_2+n\ell j}. \end{aligned}$$

□

The identity (25) is rewritten in terms of the notation of Definition 2.4 as follows.

Corollary 4.3. *Under the same notations in Proposition 4.2, we have*

$$(27) \quad \begin{aligned} & u_{c_2-nm_2}^{(d_2)} \wedge u_{c_1-nm_1}^{(d_1)} = (-q^{-1})^{\delta_{d_1=d_2}} q^\alpha u_{c_1-nm_1}^{(d_1)} \wedge u_{c_2-nm_2}^{(d_2)} \\ & + \text{sgn}(m) (-q^{-1})^{\delta_{d_1=d_2}} (q - q^{-1}) \sum_{j=\beta}^{|m_1-m_2|-\gamma} u_{c_2-nm_1-\text{sgn}(m)n_j}^{(d_1)} \wedge u_{c_1-nm_2+\text{sgn}(m)n_j}^{(d_2)}. \end{aligned}$$

Remarks. (i) Note that the identity (27) depends only on the inequality relationship between d_1 and d_2 (c_1 and c_2). It is independent of ℓ .

Let ℓ and j be integers such that $1 \leq j \leq \ell + 1$. Let

$$u = u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_2)} \wedge \cdots$$

be a q -wedge product of level ℓ . Define d'_1, d'_2, \dots as

$$d'_r = \begin{cases} d_r & \text{if } d_r < j \\ d_r + 1 & \text{if } d_r \geq j \end{cases}, \quad (r = 1, 2, \dots).$$

Then,

$$u' = u_{k_1}^{(d'_1)} \wedge u_{k_2}^{(d'_2)} \wedge \dots$$

is the q -wedge product of level $\ell + 1$. In this way, we regard a q -wedge product u of level ℓ as the q -wedge product of level $\ell + 1$.

(ii). Let $k'_1 = k_1 - c_1 + c_2 - \text{sgn}(m)n\ell j$ and $k'_2 = k_2 + c_1 - c_2 + \text{sgn}(m)n\ell j$. That is to say, $u_{k'_1} \wedge u_{k'_2}$ appears in the summation of (25). Then, k'_1 and k'_2 satisfy following properties.

- (a). k'_1 and k'_2 are in between k_1 and k_2 , i.e. $k_1 < k'_i < k_2$, ($i = 1, 2$) or $k_1 > k'_i > k_2$, ($i = 1, 2$).
- (b). k'_1 and k'_2 swap the c -part with k_1 and k_2 . That is, there exist $m'_1, m'_2 \in \mathbb{Z}$ such that $k'_1 = c_2 + n(d_1 - 1) - n\ell m'_1$ and $k'_2 = c_1 + n(d_2 - 1) - n\ell m'_2$.
- (c). $k'_1 + k'_2 = k_1 + k_2$.

In abacus presentation, the positions of k'_1, k'_2 and k_1, k_2 look like

$$\begin{array}{cc|cc} d = d_1 & & d = d_2 & \\ \vdots & \vdots & \vdots & \textcircled{k_2} \\ \vdots & \textcircled{k'_2} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \textcircled{k'_1} & \vdots \\ \textcircled{k_1} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ c = c_1 & c = c_2 & c = c_1 & c = c_2 \end{array}$$

4.2. Several properties of q -wedge products. In this paragraph, we summarize other properties of q -wedge products which will be needed in the proof of our main theorems.

Lemma 4.4 ([Ugl00]). *If $k \geq t$, then*

- (i). $u_t \wedge u_k \wedge u_{k-1} \wedge \dots \wedge u_t = 0$,
- (ii). $u_k \wedge u_{k-1} \wedge \dots \wedge u_t \wedge u_k = 0$.

More generally, we have

Corollary 4.5. *If $k \geq m \geq t$, then*

- (i). $u_m \wedge u_k \wedge u_{k-1} \wedge \dots \wedge u_t = 0$,
- (ii). $u_k \wedge u_{k-1} \wedge \dots \wedge u_t \wedge u_m = 0$.

Proof. The first assertion immediately follows from Lemma 4.4 (i). We prove (ii) by induction on $m - t$. If $m = t$, then the assertion follows from Lemma 4.4 (ii).

Let $m - t > 0$. From the identity (25), we know that there exist $b_0(q), \dots, b_{m-t}(q)$ such that

$$u_t \wedge u_m = \sum_{j=0}^{m-t} b_j(q) u_{m-j} \wedge u_{t+j}.$$

Then,

$$u_k \wedge \cdots \wedge u_{t+1} \wedge u_t \wedge u_m = \sum_{j=0}^{m-t} b_j(q) u_k \wedge \cdots \wedge u_{t+1} \wedge u_{m-j} \wedge u_{t+j}$$

Here, by the induction hypothesis, $u_k \wedge \cdots \wedge u_{t+1} \wedge u_{m-j} = 0$ for all $0 \leq j \leq m-t$. Therefore $u_k \wedge \cdots \wedge u_{t+1} \wedge u_t \wedge u_m = 0$. \square

The next corollary follows from the above corollary and Corollary 4.3.

Corollary 4.6. *If $k \geq m \geq t$ and $1 \leq j \leq \ell$, then*

- (i). $u_m^{(j)} \wedge u_k^{(j)} \wedge u_{k-1}^{(j)} \wedge \cdots \wedge u_t^{(j)} = 0$,
- (ii). $u_k^{(j)} \wedge u_{k-1}^{(j)} \wedge \cdots \wedge u_t^{(j)} \wedge u_m^{(j)} = 0$.

Definition 4.7. *Let*

$$u = u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_2)} \wedge \cdots \wedge u_{k_r}^{(d_r)} \quad , \quad k_a = c_a - nm_a \quad , \quad (a = 1, 2, \dots, r) \quad \text{and}$$

$$v = u_{g_1}^{(d'_1)} \wedge u_{g_2}^{(d'_2)} \wedge \cdots \wedge u_{g_t}^{(d'_t)} \quad , \quad g_b = c'_b - nm'_b \quad , \quad (b = 1, 2, \dots, t).$$

and suppose that $d_a \neq d'_b$ for all $a \in \{1, \dots, r\}$ and $b \in \{1, \dots, t\}$. Then we define $\xi(u, v)$ as

$$(28) \quad \xi(u, v) = \#\{(a, b) \mid c_a = c'_b, \quad u_{k_a}^{(d_a)} < u_{g_b}^{(d'_b)}\}.$$

Lemma 4.8 ([Ugl00]). *Let $a \in \mathbb{Z}$, $t \in \mathbb{Z}_{\geq 0}$, $1 \leq i \leq \ell$, and $1 \leq j \leq \ell$.*

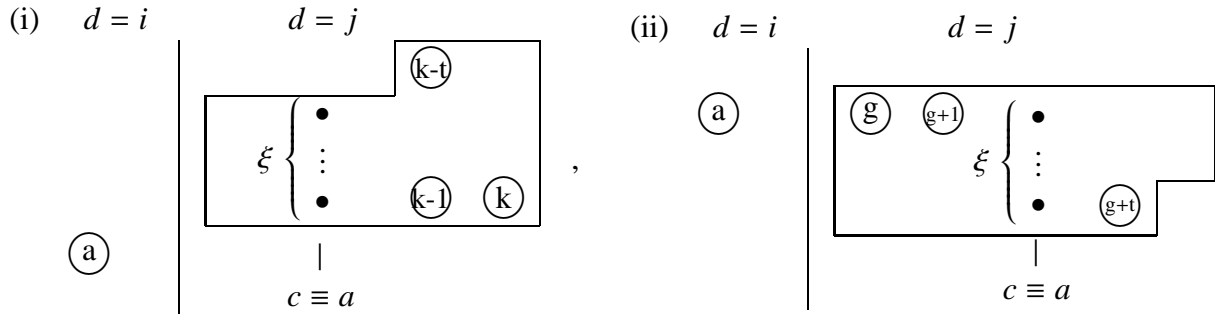
- (i). *Let $u_k^{(j)}$ be the maximal element such that $u_k^{(j)} < u_a^{(i)}$. Let $u_{[k, k-t]}^{(j)} = u_k^{(j)} \wedge u_{k-1}^{(j)} \wedge \cdots \wedge u_{k-t}^{(j)}$. Then,*

$$u_a^{(i)} \wedge u_{[k, k-t]}^{(j)} = q^{-\xi(u_{[k, k-t]}^{(j)}, u_a^{(i)})} u_{[k, k-t]}^{(j)} \wedge u_a^{(i)}.$$

- (ii). *Let $u_g^{(j)}$ be the minimal element such that $u_g^{(j)} > u_a^{(i)}$. Let $u_{[g+t, g]}^{(j)} = u_{g+t}^{(j)} \wedge u_{g+t-1}^{(j)} \wedge \cdots \wedge u_g^{(j)}$. Then,*

$$u_a^{(i)} \wedge u_{[g+t, g]}^{(j)} = q^{\xi(u_a^{(i)}, u_{[g+t, g]}^{(j)})} u_{[g+t, g]}^{(j)} \wedge u_a^{(i)}.$$

In the abacus presentation, $u_a^{(i)}$, $u_{[k, k-t]}^{(j)}$ and $u_{[g+t, g]}^{(j)}$ look as follows.



where the boxed region means that all positions are occupied by beads.

Proof. We only show (i) by induction on t . If $t = 0$, then the assertion follows from the identity (27).

Let $t \geq 1$. Then, from the identity (27), we have

$$(29) \quad u_a^{(i)} \wedge u_{k-t}^{(j)} = q^\alpha u_{k-t}^{(j)} \wedge u_a^{(i)} + \sum_{m=1}^t b_m(q) u_{k-t+m}^{(j)} \wedge u_{a-m}^{(i)},$$

where

$$\alpha = \begin{cases} -1 & \text{if } a \equiv k - t \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

(see Remark (ii) after Proposition 4.2).

Put $\xi = \xi(u_{[k,k-t]}^{(j)}, u_a^{(i)})$ and $\xi' = \xi(u_k^{(j)} \wedge u_{k-1}^{(j)} \wedge \cdots \wedge u_{k-t+1}^{(j)}, u_a^{(i)})$. Then $\xi = \xi' + \alpha$, and

$$\begin{aligned} u_a^{(i)} \wedge u_{[k,k-t]}^{(j)} &= u_a^{(i)} \wedge u_k^{(j)} \wedge u_{k-1}^{(j)} \wedge \cdots \wedge u_{k-t}^{(j)} \\ &= q^{\xi'} u_k^{(j)} \wedge u_{k-1}^{(j)} \wedge \cdots \wedge u_{k-t+1}^{(j)} \wedge u_a^{(i)} \wedge u_{k-t}^{(j)} && \text{(By the induction hypothesis)} \\ &= q^{\xi'+\alpha} u_k^{(j)} \wedge \cdots \wedge u_{k-t+1}^{(j)} \wedge u_{k-t}^{(j)} \wedge u_a^{(i)} \\ &\quad + \sum_{m=1}^t q^{\xi'} b_m(q) u_k^{(j)} \wedge \cdots \wedge u_{k-t+1}^{(j)} \wedge u_{k-t+m}^{(j)} \wedge u_{a-m}^{(i)} && \text{(By (29))} \\ &= q^\xi u_{[k,k-t]}^{(j)} \wedge u_a^{(i)} && \text{(By Corollary 4.5)} \end{aligned}$$

□

Definition 4.9. Let $1 \leq j \leq \ell$ and λ be a partition. We define

$$\lambda^{[j]} = u_{s_j+\lambda_1}^{(j)} \wedge u_{s_j+\lambda_2-1}^{(j)} \wedge u_{s_j+\lambda_3-2}^{(j)} \wedge \cdots.$$

In particular,

$$\emptyset^{[j]} = u_{s_j}^{(j)} \wedge u_{s_j-1}^{(j)} \wedge u_{s_j-2}^{(j)} \wedge \cdots.$$

Corollary 4.10. Let $1 \leq j \leq \ell$, $r > 0$, $t > 0$, and put

$$\emptyset^{[j]} = u_{s_j}^{(j)} \wedge u_{s_j-1}^{(j)} \wedge \cdots \wedge u_{s_j-r}^{(j)} \quad \text{and} \quad u = u_{g_1}^{(d_1)} \wedge u_{g_2}^{(d_2)} \wedge \cdots \wedge u_{g_t}^{(d_t)}.$$

For each $b \geq 1$, let $u_{h_b}^{(j)}$ be the minimal element such that $u_h^{(j)} > u_{g_b}^{(d_b)}$. If $d_b \neq j$ and $s_j \geq h_b \geq s_j - r$ for all $b = 1, 2, \dots, t$, then

$$u \wedge \emptyset^{[j]} = q^{\xi(u, \emptyset^{[j]}) - \xi(\emptyset^{[j]}, u)} \emptyset^{[j]} \wedge u.$$

5. PROOF OF THEOREM 3.4

We only prove Theorem 3.4 in the case of $\varepsilon = -$. The proof in the case of $\varepsilon = +$ is similar. Through this section we fix j ($1 \leq j \leq \ell$).

5.1. Preliminary for the proof. Fix an sufficiently large integer r so that for every ordered q -wedge product appearing in our argument, all of the components after r -th factor are consecutive. We are able to truncate q -wedge products at the first r parts. See [Ugl00] for detail. Then $|\lambda; s\rangle$ can be identified with v_λ defined by

$$(30) \quad |\lambda; s\rangle = v_\lambda \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots.$$

First, we extend the definition of "sufficiently large" on the finite q -wedge products and introduce some notations.

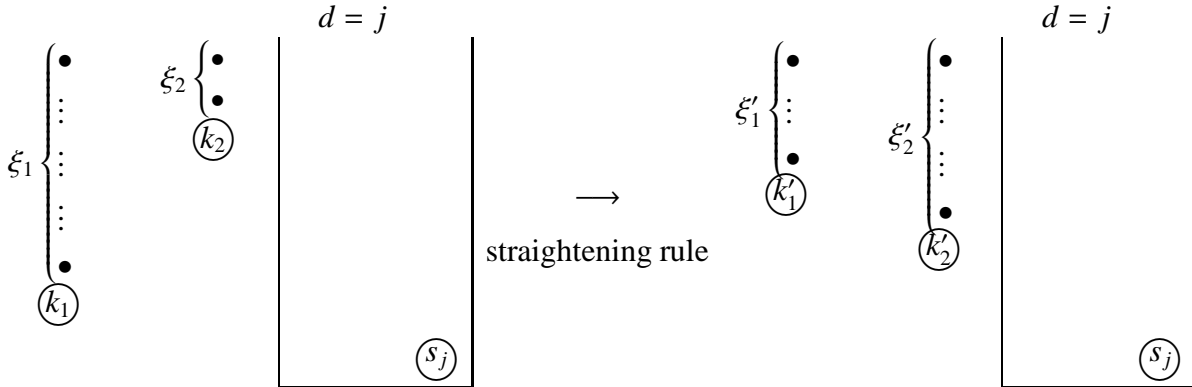
Definition 5.1. Let $u_k = u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_r}$ be an ordered q -wedge product and write $k_a = c_a + n(d_a - 1) - n\ell m_a$ for $a = 1, 2, \dots, r$ as in (4). Then define \check{u}_k to be the q -wedge obtained from u_k by removing all factors $u^{(d_a)}$ with $d_a = j$.

Lemma 5.2. Suppose that s_j is sufficiently large for $|\lambda; s\rangle$ and $\Delta_{\lambda, \mu}^-(q) \neq 0$. Let $v_\lambda = |\lambda; s\rangle$, $v_\mu = |\mu; s\rangle$ and r as above. Then,

$$\xi(\emptyset^{[j]}, \check{v}_\lambda) = \xi(\emptyset^{[j]}, \check{v}_\mu).$$

Proof. Let $u_{k_1} \wedge u_{k_2}$ be a q -wedge product. Suppose that s_j is sufficiently large for $u_{k_1} \wedge u_{k_2}$. Let $u_{k'_1} \wedge u_{k'_2}$ be a q -wedge product which appears in the linear expansion of the straightening of $u_{k_2} \wedge u_{k_1}$.

Put $\xi = \xi(\emptyset^{[j]}, u_{k_1} \wedge u_{k_2})$, $\xi_1 = \xi(\emptyset^{[j]}, u_{k_1})$, $\xi_2 = \xi(\emptyset^{[j]}, u_{k_2})$, $\xi' = \xi(\emptyset^{[j]}, u_{k'_1} \wedge u_{k'_2})$, $\xi'_1 = \xi(\emptyset^{[j]}, u_{k'_1})$ and $\xi'_2 = \xi(\emptyset^{[j]}, u_{k'_2})$ (see Definition 4.9). Note that $\xi = \xi_1 + \xi_2$ and $\xi' = \xi'_1 + \xi'_2$. Then, from the abacus presentation below, we obtain $\xi = \xi'$. That is, the straightening rule preserves ξ if s_j is sufficiently large.



where beads are filled in the boxed region.

If $\Delta_{\lambda, \mu}^-(q) \neq 0$, then v_μ appears in the linear expansion of the straightening of \overline{v}_λ . Therefore, the above argument assures the assertion. \square

From Lemma.4.8, we have

Corollary 5.3 (see [Ugl00], Lemma 5.19). *If s_j is sufficiently large for an ordered q -wedge product $u_k = u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_r}$. Then*

$$u_k = q^{-\xi(\emptyset^{[j]}, \check{u}_k)} \emptyset^{[j]} \wedge \check{u}_k.$$

Example 5.4. Let $n = 2$, $\ell = 3$, $s = (0, 2, -2)$ and $\lambda = ((1, 1), \emptyset, (3))$. Then s_2 is sufficiently large for $|\lambda; s\rangle$. Take $r = 7$, then

$$\begin{aligned}
u_k &= u_5 \wedge u_4 \wedge u_3 \wedge u_1 \wedge u_{-2} \wedge u_{-3} \wedge u_{-4} \wedge u_{-7} \\
&= u_1^{(3)} \wedge u_2^{(2)} \wedge u_1^{(2)} \wedge u_1^{(1)} \wedge u_0^{(2)} \wedge u_{-1}^{(2)} \wedge u_0^{(1)} \wedge u_{-3}^{(3)} \\
&= q^{-1} \overline{u_1^{(3)} \wedge u_2^{(2)} \wedge u_1^{(2)} \wedge u_0^{(2)} \wedge u_{-1}^{(2)}} \wedge u_1^{(1)} \wedge u_0^{(1)} \wedge u_{-3}^{(3)} \\
&= q^{-3} u_2^{(2)} \wedge u_1^{(2)} \wedge u_0^{(2)} \wedge u_{-1}^{(2)} \wedge u_1^{(3)} \wedge u_1^{(1)} \wedge u_0^{(1)} \wedge u_{-3}^{(3)} \\
&= q^{-3} \emptyset^{[2]} \wedge \check{u}_k.
\end{aligned}$$

Lemma 5.5. If s_j is sufficiently large for an ordered q -wedge product $u_k = u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_r}$. Then,

$$\overline{u_k} = q^{-\xi(\emptyset^{[j]}, \check{u}_k)} \emptyset^{[j]} \wedge \overline{\check{u}_k}.$$

Proof. Let $\xi = \xi(\emptyset^{[j]}, \check{u}_k)$ and $\eta = \xi(\check{u}_k, \emptyset^{[j]})$. By Corollary 5.3, we have

$$u_k = q^{-\xi} \emptyset^{[j]} \wedge \check{u}_k.$$

Thus, we have

$$\begin{aligned}
\overline{u_k} &= q^\xi q^{-\xi-\eta} \overline{\check{u}_k} \wedge \overline{\emptyset^{[j]}} && \text{(Definition of bar involution (8))} \\
&= q^{-\eta} \overline{\check{u}_k} \wedge \emptyset^{[j]} && (\overline{\emptyset^{[j]}} = \emptyset^{[j]}) \\
&= q^{-\eta} q^{\eta-\xi} \emptyset^{[j]} \wedge \overline{\check{u}_k} && \text{(By Corollary 4.10)} \\
&= q^{-\xi} \emptyset^{[j]} \wedge \overline{\check{u}_k}
\end{aligned}$$

□

5.2. Proof of Theorem 3.4. Let $\check{\Pi}^\ell$ be the subset of Π^ℓ whose j -th component is the empty Young diagram. i.e.

$$(31) \quad \check{\Pi}^\ell = \{\lambda \in \Pi^\ell \mid \lambda^{(j)} = \emptyset\}.$$

Theorem 3.4 is a direct consequence of the next proposition.

Proposition 5.6. Suppose that s_j is sufficiently large for $|\lambda; s\rangle$. Then,

$$G^+(\lambda; s) = \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^+(q) |\mu; s\rangle, \quad G^-(\lambda; s) = \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) |\mu; s\rangle,$$

where $\check{\lambda}$ (resp. $\check{\mu}, \check{s}$) is obtained by omitting the j -th component of λ (resp. μ, s).

Proof. We only show the statement in for G^- . The case of G^+ is treated similarly.

Take a sufficiently large integer r . Put $F = \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) |\mu; s\rangle$. We prove $\overline{F} = F$ and $F \equiv |\lambda; s\rangle \pmod{q^{-1} \mathcal{L}^-}$.

The second statement is clear since $\check{\lambda} = \check{\mu}$ if and only if $\lambda = \mu$. We show $\overline{F} = F$. Let $\xi = \xi(\emptyset^{[j]}, \check{v}_\lambda)$.

$$\begin{aligned}
\overline{F} &= \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q^{-1}) \overline{u_\mu} \\
&= \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q^{-1}) q^{-\xi} \emptyset^{[j]} \wedge \overline{u_\mu} \quad (\text{By Lemma 5.5 \& Lemma 5.2}) \\
&= q^{-\xi} \left(\sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q^{-1}) \emptyset^{[j]} \wedge \overline{u_\mu} \right) \\
&= q^{-\xi} \emptyset^{[j]} \wedge \left(\sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q^{-1}) \overline{u_\mu} \right) \\
&= q^{-\xi} \emptyset^{[j]} \wedge \overline{\left(\sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) u_\mu \right)}
\end{aligned}$$

Note that $G^-(\check{\lambda}; \check{s}) = \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) u_\mu$ and $\overline{G^-(\check{\lambda}; \check{s})} = G^-(\check{\lambda}; \check{s})$. Therefore,

$$\begin{aligned}
\overline{F} &= q^{-\xi} \emptyset^{[j]} \wedge \left(\sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) u_\mu \right) \\
&= \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) v_\mu \quad (\text{By Corollary 5.3 \& Lemma 5.2}) \\
&= F.
\end{aligned}$$

□

6. PROOF OF THEOREM 3.5

Throughout this section, we fix j ($1 \leq j \leq \ell$).

6.1. The quotient space $\widetilde{F}_q[s]_{\leq N}$. In this paragraph, we fix a positive integer N and assume that s_j is sufficiently small for N , i.e. $s_i - s_j \geq N$ for all $i \neq j$.

We define $\widetilde{F}_q[s]_{\leq N}$ to be the subspace spanned by $\{|\lambda; s\rangle \mid \lambda^{(j)} = \emptyset, |\lambda| \leq N\}$. We also define a map $\pi: F_q[s] \rightarrow \widetilde{F}_q[s]_{\leq N}$ (quotient map) by

$$\pi(|\lambda; s\rangle) = \begin{cases} |\lambda; s\rangle & \text{if } \lambda^{(j)} = \emptyset \text{ and } |\lambda| \leq N \\ 0 & \text{otherwise} \end{cases}$$

We import the bar involution on $\widetilde{F}_q[s]_{\leq N}$ from $F_q[s]$, that is

$$(32) \quad \overline{\pi(|\lambda; s\rangle)} = \pi(\overline{|\lambda; s\rangle}), \quad (|\lambda; s\rangle \in \widetilde{F}_q[s]_{\leq N}).$$

The unitriangularity of the bar involution (11) and Lemma 3.3 assure that the bar involution $\widetilde{F}_q[s]_{\leq N}$ is well-defined.

It is clear that the following two property hold from the definition of $\widetilde{F}_q[s]_{\leq N}$.

Proposition 6.1. *Let $\varepsilon \in \{+, -\}$. There is a unique basis $\{\widetilde{G}^\varepsilon(\lambda; s) \mid \lambda \in \check{\Pi}^\ell, |\lambda| \leq N\}$ of $\widetilde{F}_q[s]_{\leq N}$ such that*

$$\begin{aligned}
\text{(i)} \quad & \overline{\widetilde{G}^\varepsilon(\lambda; s)} = \widetilde{G}^\varepsilon(\lambda; s), \\
\text{(ii)} \quad & \widetilde{G}^\varepsilon(\lambda; s) \equiv |\lambda; s\rangle \pmod{q^\varepsilon \widetilde{\mathcal{L}}^\varepsilon}, \text{ where } \widetilde{\mathcal{L}}^\varepsilon = \bigoplus_{\lambda \in \check{\Pi}^\ell} \mathbb{Q}[q^\varepsilon] |\lambda; s\rangle
\end{aligned}$$

and $\check{\Pi}^\ell = \{\lambda \in \Pi^\ell \mid \lambda^{(j)} = \emptyset\}$.

Definition 6.2. Let $\varepsilon \in \{+, -\}$. Suppose that $\lambda^{(j)} = \mu^{(j)} = \emptyset$, $|\lambda| \leq N$ and $|\mu| \leq N$. Define $\widetilde{\Delta}_{\lambda, \mu}^\varepsilon(q)$ by

$$\widetilde{G}^+(\mu; s) = \sum_{\lambda \in \check{\Pi}} \widetilde{\Delta}_{\lambda, \mu}^+(q) |\lambda; s\rangle, \quad \widetilde{G}^-(\lambda; s) = \sum_{\mu \in \check{\Pi}} \widetilde{\Delta}_{\lambda, \mu}^-(q) |\mu; s\rangle.$$

Proposition 6.3. Let $\varepsilon \in \{+, -\}$. If $\lambda^{(j)} = \mu^{(j)} = \emptyset$, $|\lambda| \leq N$ and $|\mu| \leq N$, then

$$\widetilde{\Delta}_{\lambda, \mu}^\varepsilon(q) = \Delta_{\lambda, \mu}^\varepsilon(q).$$

Note that if $N \geq |\lambda|$ and $N \geq |\mu|$, then $\widetilde{\Delta}_{\lambda, \mu}^\varepsilon(q)$ is independent of the choice of N .

6.2. Proof of Theorem 3.5. As in §4, we only prove Theorem 3.5 in the case of $\varepsilon = -$.

In this paragraph, we assume that s_j is sufficiently small for $|\lambda; s\rangle$. Let $N = |\lambda|$ and we fix a sufficient large integer r .

The structure of our proof of Theorem 3.5 is similar to that of Theorem 3.4. Lemma 6.4 and Lemma 6.5 play roles similar to Lemma 5.2 and Corollary 5.3 respectively.

Lemma 6.4. Let $\lambda, \mu \in \Pi^\ell$ such that $\Delta_{\lambda, \mu}^-(q) \neq 0$. If s_j is sufficiently small for $|\lambda; s\rangle$ and $\lambda^{(j)} = \mu^{(j)} = \emptyset$, then

$$\xi(\emptyset^{[j]}, \check{v}_\lambda) = \xi(\emptyset^{[j]}, \check{v}_\mu).$$

Proof. Let $u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_2)}$ be a q -wedge product such that $d_i \neq j$ and $k_i \geq s_j$ for $i = 1, 2$. Let $u_{k'_1}^{(d'_1)} \wedge u_{k'_2}^{(d'_2)}$ be a q -wedge product which appears in the linear expansion of the straightening of $u_{k_2} \wedge u_{k_1}$.

We put $\xi = \xi(\emptyset^{[j]}, u_{k_1}^{(d_1)} \wedge u_{k_2}^{(d_2)})$, $\xi_1 = \xi(\emptyset^{[j]}, u_{k_1}^{(d_1)})$, $\xi_2 = \xi(\emptyset^{[j]}, u_{k_2}^{(d_2)})$, $\xi' = \xi(\emptyset^{[j]}, u_{k'_1}^{(d'_1)} \wedge u_{k'_2}^{(d'_2)})$, $\xi'_1 = \xi(\emptyset^{[j]}, u_{k'_1}^{(d'_1)})$ and $\xi'_2 = \xi(\emptyset^{[j]}, u_{k'_2}^{(d'_2)})$. Note that $\xi = \xi_1 + \xi_2$ and $\xi' = \xi'_1 + \xi'_2$.

Then, from the abacus presentation below, we obtain $\xi = \xi'$.

$$\begin{array}{ccc}
\begin{array}{c} d = d_1 \\ \xi_1 \left\{ \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \right. \end{array} & \begin{array}{c} d = d_2 \\ \xi_2 \left\{ \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \right. \end{array} & \begin{array}{c} d = j \\ \boxed{} \end{array} \\
\textcircled{k_2} & & \\
& \longrightarrow & \\
& & \text{straightening rule} \\
& & \textcircled{k'_1} \\
& & \textcircled{k'_2}
\end{array}$$

where beads are filled in the boxed region.

Since s_j is sufficiently small for v_λ , for each $i \neq j$,

$$\lambda_1^{(i)} + s_i > \lambda_2^{(i)} - 1 + s_i > \cdots > \lambda_l^{(i)} - l + s_i \geq s_j$$

where $l = l(\lambda^{(i)})$ is the length of $\lambda^{(i)}$.

If $\Delta_{\lambda, \mu}^-(q) \neq 0$, then v_μ appears in the linear expansion of the straightening of $\overline{v_\lambda}$. Therefore, the above argument assures the assertion. \square

Lemma 6.5. *Let $\lambda \in \Pi^\ell$. If $\lambda^{(j)} = \emptyset$, then*

$$v_\lambda = q^{-\xi(\check{v}_\lambda, \emptyset^{[j]})} \check{v}_\lambda \wedge \emptyset^{[j]}.$$

See Definition 4.7, Definition 5.1 and Definition 4.9 for the definition of ξ , \check{v}_λ and $\emptyset^{[j]}$ respectively.

Proof. The proof follows from Lemma 4.8 and Lemma 6.4. (see also Example 5.4.) \square

Lemma 6.6. *Let $1 \leq j \leq \ell$, $m > 0$ and λ be a partition of length at most m . Let $1 \leq d \leq \ell$ and k be a integer satisfying $\lambda_1 + s_j \leq k$. If $j \neq d$, then $\lambda^{[j]} \wedge u_k^{(d)}$ is expanded as*

$$\lambda^{[j]} \wedge u_k^{(d)} = q^{-\xi(u_k^{(d)}, \lambda^{[j]}) + \xi(\lambda^{[j]}, u_k^{(d)})} u_k^{(d)} \wedge \lambda^{[j]} + \sum_{|\mu| > |\lambda|} b_\mu(q) u_{k-|\mu|+|\lambda|}^{(d)} \wedge \mu^{[j]},$$

where $\lambda^{[j]}$ is defined in Definition 4.9.

Moreover, if $b_\mu(q) \neq 0$, then $\mu_1 + s_j \leq k$.

Proof. Applying the identity (27) repeatedly, we expand $\lambda^{[j]} \wedge u_k^{(d)}$ as a linear combination of $u_{k'}^{(d)} \wedge \mu^{[j]}$ such that $k' \leq k$. \square

Corollary 6.7. *Let $1 \leq j \leq \ell$, $m > 0$, $t > 0$,*

$$\emptyset^{[j]} = u_{s_j}^{(j)} \wedge u_{s_j-1}^{(j)} \wedge \cdots \wedge u_{s_j-m}^{(j)} \quad \text{and} \quad u = u_{g_1}^{(d_1)} \wedge u_{g_2}^{(d_2)} \wedge \cdots \wedge u_{g_t}^{(d_t)}.$$

If $d_b \neq j$ for all $b = 1, 2, \dots, t$ and $s_j - r \leq g_1 \leq g_2 \leq \cdots \leq g_t$, then $\emptyset^{[j]} \wedge u$ can be written in the form

$$\emptyset^{[j]} \wedge u = q^{-\xi(u, \emptyset^{[j]}) + \xi(\emptyset^{[j]}, u)} u \wedge \emptyset^{[j]} + \sum_{\mu \neq \emptyset} v_\mu(q) \wedge \mu^{[j]},$$

where $v_\mu(q)$ is a linear combination of q -wedge products.

Proof. Apply Lemma 6.6 repeatedly. \square

In the proof of Theorem 3.5, the next two lemmas (Lemma 6.8 and Lemma 6.9) play roles similar to Corollary 4.10 and Lemma 5.5 in the proof of Theorem 3.4.

Lemma 6.8. *Let $\lambda \in \Pi^\ell$. If $\lambda^{(j)} = \emptyset$, then*

$$\pi(\emptyset^{[j]} \wedge \overline{\check{v}_\lambda}) = q^{-\xi(\check{v}_\lambda, \emptyset^{[j]}) + \xi(\emptyset^{[j]}, \check{v}_\lambda)} \pi(\check{v}_\lambda \wedge \emptyset^{[j]}).$$

Proof. Let $\xi = \xi(\emptyset^{[j]}, \check{v}_\lambda)$, $\eta = \xi(\check{v}_\lambda, \emptyset^{[j]})$ and

$$\check{v}_\lambda = u_{g_1}^{(d_1)} \wedge u_{g_2}^{(d_2)} \wedge \cdots \wedge u_{g_t}^{(d_t)}.$$

From the definition of the bar involution (8),

$$\overline{\check{v}_\lambda} = (-q)^{\kappa(d)} q^{-\kappa(c)} u_{g_t}^{(d_t)} \wedge u_{g_{t-1}}^{(d_{t-1})} \wedge \cdots \wedge u_{g_1}^{(d_1)},$$

where $(-q)^{\kappa(d)}$ and $q^{-\kappa(c)}$ are suitable constants (see (8)). Then, from Corollary 6.7,

$$\begin{aligned} \emptyset^{[j]} \wedge \overline{\check{v}_\lambda} &= (-q)^{\kappa(d)} q^{-\kappa(c)} \emptyset^{[j]} \wedge u_{g_t}^{(d_t)} \wedge u_{g_{t-1}}^{(d_{t-1})} \wedge \cdots \wedge u_{g_1}^{(d_1)} \\ &= (-q)^{\kappa(d)} q^{-\kappa(c)} q^{\eta-\xi} u_{g_t}^{(d_t)} \wedge u_{g_{t-1}}^{(d_{t-1})} \wedge \cdots \wedge u_{g_1}^{(d_1)} \wedge \emptyset^{[j]} + \sum_{\mu \neq \emptyset} v_\mu(q) \wedge \mu^{(j)} \\ &= q^{\eta-\xi} \overline{\check{v}_\lambda} \wedge \emptyset^{[j]} + \sum_{\mu \neq \emptyset} v_\mu(q) \wedge \mu^{[j]}, \end{aligned}$$

where $v_\mu(q)$ is a linear combination of q -wedge products.

Finally, we shall prove $\pi(v_\mu(q) \wedge \mu^{[j]}) = 0$ if $\mu \neq \emptyset$. To do it, it is enough to prove the next claim.

Claim . Let $\mu \neq \emptyset$ and $\mathbf{v} \in \Pi^\ell$ such that $\mathbf{v}^{(j)} = \emptyset$. Then,

$$\pi(\check{u}_\mathbf{v} \wedge \mu^{[j]}) = 0.$$

(Proof of Claim)

Define $\mathbf{v}_\mu \in \Pi^\ell$ as $\mathbf{v}_\mu^{(j)} = \mu$ and $\mathbf{v}_\mu^{(i)} = \mathbf{v}^{(i)}$ ($i \neq j$). From the straightening rule ((25) or (27)), any $|\rho; s\rangle$ appearing in the linear expansion of the straightening of $\check{u}_\mathbf{v} \wedge \mu^{[j]}$ is less than or equal to $|\mathbf{v}_\mu; s\rangle$. Thus, from Lemma 3.3, the j -th component is not empty. \square

Lemma 6.9. Let $\lambda \in \Pi^\ell$. If $\lambda^{(j)} = \emptyset$, then

$$\pi(\overline{v_\lambda}) = q^{-\xi(\check{v}_\lambda, \emptyset^{[j]})} \pi(\overline{\check{v}_\lambda} \wedge \emptyset^{[j]}).$$

Proof. The proof of this proposition is similarly argued to the proof of Lemma 5.5.

Let $\xi = \xi(\emptyset^{[j]}, u_g)$ and $\eta = \xi(u_g, \emptyset^{[j]})$. Then,

$$\begin{aligned} \overline{v_\lambda} &= \overline{q^{-\xi} \check{v}_\lambda \wedge \emptyset^{[j]}} \\ &= q^\xi q^{-\xi-\eta} \overline{\emptyset^{[j]}} \wedge \overline{\check{v}_\lambda} \\ &= q^{-\eta} \emptyset^{[j]} \wedge \overline{\check{v}_\lambda}. \end{aligned}$$

Thus, from Lemma 6.8,

$$\pi(\overline{v_\lambda}) = q^{-\eta} q^{\eta-\xi} \pi(\overline{\check{v}_\lambda} \wedge \emptyset^{[j]}) = q^{-\xi} \pi(\overline{\check{v}_\lambda} \wedge \emptyset^{[j]}).$$

\square

Now Theorem 3.5 is an immediate consequence of the next proposition and Proposition 6.3.

Proposition 6.10. *Let $\lambda \in \check{\Pi}^\ell$. Suppose that s_j is sufficiently small for $|\lambda; s\rangle$. Then,*

$$\widetilde{G}^+(\lambda; s) = \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^+(q) \pi(|\mu; s\rangle) \quad , \quad \widetilde{G}^-(\lambda; s) = \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) \pi(|\mu; s\rangle) \quad ,$$

where $\check{\lambda}$ (resp. $\check{\mu}, \check{s}$) is obtained by omitting the j -th component of λ (resp. μ, s).

In particular,

$$\Delta_{\check{\lambda}, \check{\mu}; \check{s}}^+(q) = \widetilde{\Delta}_{\lambda, \mu; s}^+(q) \quad , \quad \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) = \widetilde{\Delta}_{\lambda, \mu; s}^-(q).$$

Proof. The proof of this proposition is similarly to that of Proposition 5.6.

We only show the statement in the case of G^- . The case of G^+ is treated similarly.

Take a sufficiently large r . Put $F = \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) \pi(|\mu; s\rangle)$. We prove $\overline{F} = F$ and $F \equiv |\lambda; s\rangle \bmod q^{-1}\mathcal{L}^-$.

The second statement is clear since $\check{\lambda} = \check{\mu}$ if and only if $\lambda = \mu$. We show $\overline{F} = F$. Let $\xi = \xi(\check{\nu}_\lambda, \emptyset^{[j]})$.

$$\begin{aligned} \overline{F} &= \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q^{-1}) \overline{\pi(u_\mu)} \\ &= \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q^{-1}) \pi(\overline{u_\mu}) \quad (\text{By the definition of bar involution for } \widetilde{F}_q[s]_{\leq N}) \\ &= \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q^{-1}) q^{-\xi} \pi(\overline{u_\mu} \wedge \emptyset^{[j]}) \quad (\text{By Lemma 6.9 \& Lemma 6.4}) \\ &= q^{-\xi} \left(\sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q^{-1}) \pi(\overline{u_\mu} \wedge \emptyset^{[j]}) \right) \\ &= q^{-\xi} \pi \left(\left(\sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q^{-1}) \overline{u_\mu} \right) \wedge \emptyset^{[j]} \right) \\ &= q^{-\xi} \pi \left(\overline{\left(\sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) u_\mu \right)} \wedge \emptyset^{[j]} \right) \end{aligned}$$

Note that $G^-(\check{\lambda}; \check{s}) = \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) u_\mu$ and $\overline{G^-(\check{\lambda}; \check{s})} = G^-(\check{\lambda}; \check{s})$. Therefore,

$$\begin{aligned} \overline{F} &= q^{-\xi} \pi \left(\left(\sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) u_\mu \right) \wedge \emptyset^{[j]} \right) \\ &= \sum_{\mu \in \check{\Pi}^\ell} \Delta_{\check{\lambda}, \check{\mu}; \check{s}}^-(q) \pi(v_\mu) \quad (\text{By Corollary 6.5 \& Lemma 6.4}) \\ &= F. \end{aligned}$$

□

REFERENCES

- [Ari] S. Ariki, *Graded q -Schur algebras*, mathArXiv 0903.3453.
- [CM] Chuang and H. Miyachi, *Hidden Hecke Algebras and Duality*, in preparation.

- [JMMO91] M. Jimbo, K. C. Misra, T. Miwa, and M. Okado, *Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}(n)})$ at $q = 0$* , Comm. Math. Phys. **136** (1991), no. 3, 543–566. MR1099695 (93a:17015)
- [KT02] M. Kashiwara and T. Tanisaki, *Parabolic Kazhdan-Lusztig polynomials and Schubert varieties*, J. Algebra **249** (2002), no. 2, 306–325, DOI 10.1006/jabr.2000.8690. MR1901161 (2004a:14049)
- [Rou05] R. Rouquier, *Representations of rational Cherednik algebras*, Infinite-dimensional aspects of representation theory and applications, Contemp. Math., vol. 392, Amer. Math. Soc., Providence, RI, 2005, pp. 103–131. MR2189874 (2007d:20006)
- [Rou08] ———, *q -Schur algebras and complex reflection groups*, Mosc. Math. J. **8** (2008), no. 1, 119–158, 184. MR2422270 (2010b:20081)
- [Sha] P. Shan, *Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras*, math.arXiv:0811.4549.
- [SW09] T. Shoji and K. Wada, *Product formulas for the cyclotomic v -Schur algebra and for the canonical bases of the Fock space*, J. Algebra **321** (2009), no. 11, 3527–3549, DOI 10.1016/j.jalgebra.2008.03.011. MR2510060 (2010m:20074)
- [VV99] M. Varagnolo and E. Vasserot, *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. **100** (1999), no. 2, 267–297, DOI 10.1215/S0012-7094-99-10010-X. MR1722955 (2001c:17029)
- [VV08] ———, *Cyclotomic double affine Hecke algebras and affine parabolic category \mathcal{O}, I* , math.arXiv:0810.5000 (2008).
- [Ugl00] D. Uglov, *Canonical bases of higher-level q -deformed Fock spaces and Kazhdan-Lusztig polynomials*, Physical combinatorics (Kyoto, 1999), Progr. Math., vol. 191, Birkhäuser Boston, Boston, MA, 2000, pp. 249–299. MR1768086 (2001k:17030)
- [Yvo06] X. Yvonne, *A conjecture for q -decomposition matrices of cyclotomic v -Schur algebras*, J. Algebra **304** (2006), no. 1, 419–456, DOI 10.1016/j.jalgebra.2006.03.048. MR2256400 (2008d:16051)

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, CHIKUSA-KU, NAGOYA 464-8602, JAPAN

E-mail address: kazuto.iijima@math.nagoya-u.ac.jp